

The influence of the Lande g -factor in the classical general relativistic description of atomic and subatomic systems

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Abstract. We study the electromagnetic and gravitational fields of the proton and electron in terms of the Einstenian gravity via the introduction of an arbitrary Lande g -factor in the Kerr-Newman solution. We show that at length scales of the order of the reduced Compton wavelength, corrections from different values of the g -factor are not negligible and discuss the presence of general relativistic effects in highly ionized heavy atoms. On the other hand, since at the Compton-wavelength scale the gravitational field becomes spin dominated rather than mass dominated, we also point out the necessity of including angular momentum as a source of corrections to Newtonian gravity in the quantum description of gravity at this scale.

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1. Introduction

Since the construction of a well-defined and well-established quantum theory for gravity remains still as an open problem, questionings about quantum mechanical corrections to Newton's gravitational and Coulomb's electrostatic potentials at subatomic scales are completely legitimate[‡]. In this respect, corrections in the framework of causal perturbation theory [1] including one-loop contributions to graviton self-energy [2–8], suggest that quantum corrections could be relevant at length scales of the order of the reduced Compton wavelength $\lambda_C = \hbar/mc$ for the electrostatic potential [2] and at the order of the Planck length $l_P = \sqrt{G\hbar/c^3}$ for the gravitational potential [3,4].

[‡] Abusing terminology, hereafter we will refer to the Coulomb and Newton potentials as classical potentials

These questionings are also valid from a general relativistic point of view, *i.e.*, could general relativistic effects be relevant in the description of atomic and subatomic systems? This conundrum has been addressed by Martin and Pritchett in a seminal work about the role of the magnetic dipole in the gravitational field of the electron [9] and subsequently by several authors [10–19]. Additionally, Carter [20] and Rosquist [19] pointed out that general relativistic effects become relevant at the reduced Compton wavelength scale for Coulomb’s electrostatic potential while Newman *et.al* [21] found that these effects become important at the Planck length scale for Newton’s gravitational potential. However, these estimations are based on Kerr-Newman’s solution [21], which has gyromagnetic ratio $g_{\text{KN}} = 2$ [20]; albeit, even for the electron, the real value of the gyromagnetic ratio differs from 2, $g_e = 2.0023193043768(86)$ [22], and for the proton the real value is *ca.* three times g_{KN} , $g_p = 5.585694713(46)$ [22]. For this reason, a more general model which allows to study corrections from $g \neq 2$ is desirable.

In order to get an idea of how large the effects of an arbitrary g -factor may be for an extended spinning particle, we use as a toy model the simplest generalization of the Kerr-Newman solution derived by Manko [23]. We introduce the gyromagnetic ratio as a free parameter in [23] and after an asymptotic expansion of the full solution, we find that corrections to the electric field of the electron presented by Rosquist [19] using the Kerr-Newman solution, remain practically unaffected by using the real value of the g -factor. However, this is not longer true for the proton fields which certainly differ from the Kerr-Newman based description.

At this point, two natural questions arise: *i*) If both theories, quantum mechanics and general relativity, predict corrections at the same length scale, which one is stronger than the other? and *ii*) could be possible to detect these corrections, e.g., in atomic or subatomic systems? Concerning the first one: We compare our results with those obtained in Refs. [5, 6, 8] for the Coulomb and Newton potentials at length scales of the reduced Compton wavelength of the proton and electron, and find that corrections from general relativity are stronger than the ones given by the quantum approach for gravity. Concerning the second one: We introduce our corrections in Bohr’s description of hydrogen-like atoms and since atomic radii are far away from the region where deviations from the classical treatment are expected, we just find that corrections in this case are below the uncertainty of the calculations§.

We have organized the paper as follows. In section 2 some remarks about the solution by Manko [23] are presented and the g -factor is introduced. In section 3, we present the asymptotic expansion of the electromagnetic potentials of the solution by Manko. In section 4, for the case of the electron and proton, deviations of the electromagnetic and gravitational fields using the real value of the g -factor are analyzed, the possibility of correction in atomic systems is also discussed; we close this section with the comparison of our results with those obtained from causal perturbation theory [6, 8]. Finally, we present some concluding remarks in section 5.

§ Taking into account the uncertainties for the fundamental constants

2. General relativistic description of a rotating charged magnetic dipole

The simplest metric describing the geometry of spacetime around a stationary and axisymmetric source is given by Papapetrou's line-element

$$ds^2 = -F(dt - \omega d\phi)^2 + F^{-1} [e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (1)$$

where the metric functions F , γ and ω depend only on the Weyl-Papapetrou quasi-cylindrical coordinates ρ and z . The associated Einstein-Maxwell field equations defining the metric functions in (1) can be reformulated in terms of the Ernst complex potentials $\mathcal{E}(\rho, z)$ and $\Phi(\rho, z)$ (see [24] for details). With the aid of Sibgatullin's integral method [25, 26], the Ernst potentials can be calculated from specified axis data $e(z) := \mathcal{E}(z, \rho = 0)$ and $f(z) := \Phi(z, \rho = 0)$, by the integrals $\mathcal{E}(z, \rho) = \frac{1}{\pi} \int_{-1}^1 \frac{e(\xi)\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}}$, and $\Phi(z, \rho) = \frac{1}{\pi} \int_{-1}^1 \frac{f(\xi)\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}}$. The unknown function $\mu(\sigma)$ must satisfy the singular integral equation $\int_{-1}^1 \frac{\mu(\sigma)[e(\xi)+\bar{e}(\eta)+2f(\xi)\bar{f}(\eta)]d\sigma}{(\sigma-\tau)\sqrt{1-\sigma^2}} = 0$, and the normalization condition $\int_{-1}^1 \frac{\mu(\sigma)d\sigma}{\sqrt{1-\sigma^2}} = \pi$, where $\xi = z + i\rho\sigma$, $\eta = z + i\rho\tau$ and $\sigma, \tau \in [-1, 1]$, $\bar{e}(\eta) := [e(\eta^*)]^*$, $\bar{f}(\eta) := [f(\eta^*)]^*$ and the star stands for complex conjugation.

For a rotating charged magnetic dipole, Ernst's potentials on the symmetry axis can be taken as

$$e(z) = \frac{z - m - ia}{z + m - ia}, \quad f(z) = \frac{qz + ib}{z(z + m - ia)}. \quad (2)$$

For this case, the Ernst potentials and the corresponding metric functions were derived by Manko in [23]. However, the original paper has some minor typos and we consider appropriate to write the full expressions in this paper in order to enable the solution for further studies^{||}. The Ernst potentials of the solution are given by

$$\mathcal{E} = \frac{A - B}{A + B}, \quad \Phi = -\frac{C}{A + B}, \quad (3)$$

with

$$\begin{aligned} A &= \kappa_-^2 [(\kappa_+^2 - aq - b)(R_- r_- + R_+ r_+) + i\kappa_+(a + q)(R_- r_- - R_+ r_+)] \\ &\quad + \kappa_+^2 [(\kappa_-^2 + aq + b)(R_- r_+ + R_+ r_-) + i\kappa_-(a - q)(R_- r_+ - R_+ r_-)] - 4b(aq + b)(R_- R_+ + r_- R_+), \\ B &= m\kappa_+ \kappa_- \{ (m^2 - a^2 - q^2)(r_- + r_+ - R_- - R_+) + \kappa_+ \kappa_- (r_- + r_+ + R_- + R_+) \\ &\quad + iq[(\kappa_+ + \kappa_-)(r_- - r_+) + (\kappa_+ - \kappa_-)(R_+ - R_-)] \}, \\ C &= \kappa_+ \kappa_- \{ [q(m^2 - a^2 - q^2) - 2ab](R_- + R_+ - r_- - r_+) - q\kappa_+ \kappa_- (R_- + R_+ + r_+ - r_-) \\ &\quad + i[\kappa_+(q^2 + b)(R_- - R_+ + r_+ - r_-) + \kappa_-(q^2 - b)(R_+ - R_- + r_+ - r_-)] \}, \\ R_{\pm} &= \sqrt{\rho^2 + [z \pm \frac{1}{2}(\kappa_+ + \kappa_-)]}, \quad r_{\pm} = \sqrt{\rho^2 + [z \pm \frac{1}{2}(\kappa_+ - \kappa_-)]}, \quad \kappa_{\pm} = \sqrt{m^2 - a^2 - q^2 \pm 2b}. \end{aligned} \quad (4)$$

The metric functions F , γ , and ω entering into (1) are given by

$$F = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{(A + B)(\bar{A} + \bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B} + C\bar{C}}{16[(m^2 - a^2 - q^2)^2 - 4b^2]^2 R_- R_+ r_- r_+}, \quad (5)$$

$$\omega = \frac{\text{Re}[CD + i[(E - qC)(\bar{A} + \bar{B}) + mG(A + B)]]}{A\bar{A} - B\bar{B} + C\bar{C}}, \quad (6)$$

^{||} We thank Prof. Manko for providing a rectified version of the metric function ω .

with

$$\begin{aligned}
D &= m\{\kappa_-^2[(b-q^2)(R_-r_- + R_+r_+) - iq\kappa_+(R_-r_- - R_+r_+)] + \kappa_+^2[(b+q^2)(R_-r_+ + R_+r_-) \\
&\quad - iq\kappa_-(R_-r_+ - R_+r_-)] - 2b[(m^2 - a^2 + q^2)(R_-R_+ + r_+r_+) + \kappa_- \kappa_+(R_-R_+ - r_-r_+) + 2\kappa_-^2\kappa_+^2]\} \\
&\quad + \kappa_- \kappa_+ \{[b(3m^2 + a^2 + q^2 - 2iaz) - q(m^2 - a^2 - q^2)(a - iz)](r_- - R_- + r_+ - R_+) \\
&\quad - (b + aq - iaz)[\kappa_- \kappa_+(R_- + R_+ + r_- + r_+) - iq(\kappa_+ + \kappa_-)(r_- - r_+) \\
&\quad - iq(\kappa_+ - \kappa_-)(R_+ - R_-)] + bz[(\kappa_+ - \kappa_-)(r_- - r_+) + (\kappa_+ + \kappa_-)(R_+ - R_-)]\}, \\
E &= \kappa_-^2\{[\kappa_+^2(m + iq) - (b + aq)(m - ia)](R_-r_- + R_+r_+) + \kappa_+[m^2 - q^2 + b + im(a + q)](R_-r_- - R_+r_+)\} \\
&\quad + \kappa_+^2\{[\kappa_-^2(m - iq) + (b + aq)(m - ia)](R_-r_+ + R_+r_-) + \kappa_-[m^2 - q^2 - b + im(a - q)](R_-r_+ - R_+r_-)\} \\
&\quad - 4b(m - ia)(b + aq)(R_-R_+ + r_-r_+) \\
&\quad + \kappa_- \kappa_+ \{[(m^2 - a^2 + q^2 - ima)(R_- + R_+ + r_- + r_+) - 2aqb](R_- - r_- + R_+ - r_+) \\
&\quad - \kappa_- \kappa_+(mz - m^2 + q^2 - ima)(R_- + R_+ + r_- + r_+) \\
&\quad + iq(mz - m^2 + q^2 - ima)[(\kappa_+ - \kappa_-)(R_- - R_+) + (\kappa_+ + \kappa_-)(r_+ - r_-)] \\
&\quad + iqb[(\kappa_+ + \kappa_-)(R_- - R_+) + (\kappa_+ - \kappa_-)(r_+ - r_-)]\}, \\
G &= \kappa_- \kappa_+ \{[(m^2 - a^2 - q^2)(z + ia) - 2iqb](r_- - R_- + r_+ - R_+) + \kappa_- \kappa_+(z + ia)(R_- + R_+ + r_- + r_+) \\
&\quad + (b + aq - iqz)[(\kappa_+ + \kappa_-)(r_- - r_+) + (\kappa_+ - \kappa_-)(R_+ - R_-)]\} \\
&\quad + m\{\kappa_+^2[(\kappa_- - iq)R_+r_- - (\kappa_- + iq)R_-r_+] + \kappa_-^2[(\kappa_+ + iq)R_+r_+ - (\kappa_+ - iq)R_-r_-] \\
&\quad + 4iqb(R_+R_- + r_+r_-)\}.
\end{aligned} \tag{7}$$

Note that F, ω , and γ satisfy $\omega \rightarrow 0$, $F \rightarrow 1$ and $\gamma \rightarrow 0$ at infinity.

3. Asymptotic expansion of gravitational and the electromagnetic potentials

In order to understand the physical meaning of the four arbitrary parameters m, a, q and b in (2) and to calculate the approximate potentials it is helpful to change the potentials \mathcal{E} and Φ to the potentials ξ and ς —which are related to the gravitational and electrostatic potentials in a very direct way (see Eqs. (17) and (19))—throughout the following definitions:

$$\mathcal{E} = \frac{1 - \xi}{1 + \xi}, \quad \Phi = \frac{\varsigma}{1 + \xi}. \tag{8}$$

These potentials satisfy [27]

$$(\xi\xi^* - \varsigma\varsigma^* - 1)\nabla^2\xi = 2(\xi^*\nabla\xi - \varsigma^*\nabla\varsigma) \cdot \nabla\xi, \tag{9}$$

$$(\xi\xi^* - \varsigma\varsigma^* - 1)\nabla^2\varsigma = 2(\xi^*\nabla\xi - \varsigma^*\nabla\varsigma) \cdot \nabla\varsigma. \tag{10}$$

Since this set of equations denotes an alternative representation of the Einstein-Maxwell field equations, they could be understood as the generalization of Laplace's equation for (1).

In order to measure the moments of an asymptotically flat spacetime, according to the Geroch-Hansen procedure [28, 29], we map the initial 3-metric to a conformal one $h_{ij} \rightarrow \tilde{h}_{ij} = \Omega^2 h_{ij}$. The conformal factor Ω should satisfy the following conditions: $\Omega|_\Lambda = \tilde{D}_i \Omega|_\Lambda = 0$ and $\tilde{D}_i \tilde{D}_j \Omega|_\Lambda = 2h_{ij}|_\Lambda$, where Λ is the point added to the initial manifold that represents infinity. Ω transforms the

potentials ξ and ζ into $\tilde{\xi} = \Omega^{-1/2}\xi$ and $\tilde{\zeta} = \Omega^{-1/2}\zeta$, respectively. The conformal factor is given by $\Omega = \bar{r}^2 = \bar{\rho}^2 + \bar{z}^2$, and the transformation relation between the barred and unbarred variables is given as

$$\bar{\rho} = \frac{\rho}{\rho^2 + z^2}, \quad \bar{z} = \frac{z}{\rho^2 + z^2}, \quad \bar{\phi} = \phi, \quad (11)$$

which brings infinity at the origin of the axes $(\bar{\rho}, \bar{z}) = (0, 0)$. The potentials $\tilde{\xi}$ and $\tilde{\zeta}$ can be written in a power series expansion of $\bar{\rho}$ and \bar{z} as

$$\tilde{\xi} = \sum_{i,j=0}^{\infty} a_{ij} \bar{\rho}^i \bar{z}^j, \quad \tilde{\zeta} = \sum_{i,j=0}^{\infty} b_{ij} \bar{\rho}^i \bar{z}^j. \quad (12)$$

Due to the analyticity of the potentials at the axis of symmetry, a_{ij} and b_{ij} must vanish when i is odd [27, 30]. The coefficients in the above power series can be calculated by the relation [27]

$$\begin{aligned} (r+2)^2 a_{r+2,s} = & -(s+2)(s+1) a_{r,s+2} + \sum_{k,l,m,n,p,g} (a_{kl} a_{mn}^* - b_{kl} b_{mn}^*) \\ & \times [a_{pg}(p^2 + g^2 - 4p - 5g - 2pk - 2gl - 2) \\ & + a_{p+2,g-2}(p+2)(p+2-2k) + a_{p-2,g+2}(g+2)(g+1-2l)] \end{aligned} \quad (13)$$

and

$$\begin{aligned} (r+2)^2 b_{r+2,s} = & -(s+2)(s+1) b_{r,s+2} + \sum_{k,l,m,n,p,g} (a_{kl} a_{mn}^* - b_{kl} b_{mn}^*) \\ & \times [b_{pg}(p^2 + g^2 - 4p - 5g - 2pk - 2gl - 2) \\ & + b_{p+2,g-2}(p+2)(p+2-2k) + b_{p-2,g+2}(g+2)(g+1-2l)], \end{aligned} \quad (14)$$

where $m = r - k - p$, $0 \leq k \leq r$, $0 \leq p \leq r - k$ with k and p even, and $n = s - l - g$, $0 \leq l \leq s + 1$, and $-1 \leq g \leq s - l$. These recursive relations could build the whole power series of $\tilde{\xi}$ and $\tilde{\zeta}$ from their values on the axis of symmetry

$$\tilde{\xi}(\bar{\rho} = 0) = \sum_{i=0}^{\infty} m_i \bar{z}^i, \quad \tilde{\zeta}(\bar{\rho} = 0) = \sum_{i=0}^{\infty} q_i \bar{z}^i. \quad (15)$$

The values of the multipolar moments of the spacetime are determined in terms of the coefficients in the series expansion (15). The relations between a_{ij} and b_{ij} can be used in order to express the moments in terms of $q_i \equiv b_{0i}$ and $m_i \equiv a_{0i}$.

Following the previous procedure, the multipolar moments for Manko's solution (in agreement with [23]) are $m_0 = m$, $m_1 = ima$, $m_2 = -ma^2$, $m_3 = -ima^3$, $q_0 = q$, $q_1 = i(b+aq)$, $q_2 = -a(aq+b)$ and $q_3 = -ia^2(aq+b)$, whence it follows that parameters m , a and q represent, the total mass, the total angular momentum per unit mass and the total charge, respectively. The parameter b is related to the magnetic dipole μ by means of $\mu = \text{Im}(q_1) = b + aq$. Then, by defining

$$b = aq \left(\frac{g}{2} - 1 \right), \quad (16)$$

with g the Lande factor one can see that the multipolar moment $\mu = \text{Im}(q_1)$ reduces to the classical expression for the magnetic dipole $\mu = \frac{gqJ}{2m}$ [31]. This simple transformation lets us study particles with an arbitrary g -factor, but restricted to the case of charged particles. An alternative approach to solve this limitation is to consider a more general solution, *e.g.* [32, 33], but due to the cumbersome

form of the expressions this will be study elsewhere. It is important to notice that if we set $b = 0$ in (2) or $g = 2$ in (16), we obtain the Kerr-Newman solution [21]. This fact shows that the asymptotic expansion of Manko's solution represents the simplest model for studying systems with arbitrary intrinsic magnetic dipole as atomic nuclei or other subatomic systems.

On the other hand, by performing the asymptotic expansion (12) in polar spherical coordinates with $i \leq 4$ and $j \leq (3 - i)$, *i.e.* up to the first term including short range interactions, we get the approximate expressions for the mass and the rotation potentials which are given, respectively, by the real and imaginary parts of ξ

$$\xi = \phi_M + i\phi_J, \quad (17)$$

$$\phi_M/m = \left[\frac{1}{r} - \frac{a^2}{r^3} P_2 \right] + \left[\frac{q^2 - m^2}{3r^3} (1 - P_2) \right] + \mathcal{O}(r^5), \quad (18)$$

$$\phi_J/J = \left[\frac{1}{r^2} + \frac{3(a^2 - m^2 + q^2)}{5r^4} \right] P_1 - \left[\frac{3(a^2 - m^2 + q^2)}{5r^4} \right] P_3 + \mathcal{O}(r^6),$$

and the approximate expressions for the electrostatic and the magnetic potentials which are given by the real and imaginary parts of ς , respectively,

$$\varsigma = \phi_E + i\phi_H, \quad (19)$$

$$\phi_E/q = \left[\frac{1}{r} - \frac{ga^2}{2r^3} P_2 \right] + \left[\frac{q^2 - m^2}{3r^3} (1 - P_2) \right] + \mathcal{O}(r^5), \quad (20)$$

$$\phi_H/\mu = \left[\frac{1}{r^2} + \frac{3(a^2 - m^2 + q^2)}{5r^4} \right] P_1 - \left[\frac{3(a^2 - m^2 + q^2)}{5r^4} \right] P_3 + \mathcal{O}(r^6),$$

Here, $P_n = P_n(\cos \theta)$ denotes the Legendre polynomial of n th degree. It is worth mentioning that although potentials ξ and ς given in (17) and (19) satisfy (9) and (10), up to the same order of approximation, their real parts do not satisfy the Laplace equation in flat space. In the classical theory, static electric and magnetic fields do not interact, however in Einstein-Maxwell theory they do, and except in very special cases, there is an interaction tending to introduce a rotation into the spacetime [34–36]. Expressions above confirm this fact in the post-Newtonian limit. In addition, it should be pointed out that the effect of a Lande g -factor different from g_{KN} will be reflected only in the electric potential.

4. Some consequences of the arbitrary Lande g -factor

Given that the magnetic dipole for Manko's solution is referred to as $\mu = gqa/2$, let us assume that the intrinsic spin angular momentum s of any charged lepton, nucleon or nuclei, can be identified with the angular momentum J of Manko's solution. In the particular case of spin-1/2 particles, the following relation holds

$$a = \frac{J}{m} = \frac{\hbar}{2m}. \quad (21)$$

By using the fact that in natural units, *e.g.*, for the electron $a_e \sim 1.93 \times 10^{-11} \text{cm} \gg q_e \sim 1.38 \times 10^{-34} \text{cm} \gg m_e \sim 6.76 \times 10^{-56} \text{cm}$, without lost of generality, the approximate expressions

for the gravitational and electrostatic potentials (18,20) can be written as

$$\phi_M = \frac{Gm}{r} \left[1 + \frac{a^2}{c^2 r^2} P_2(\cos \theta) \right], \quad \phi_E = \frac{kq}{r} \left[1 + \frac{ga^2}{2c^2 r^2} P_2(\cos \theta) \right], \quad (22)$$

where G denotes the Newton gravitational constant, k the Coulomb constant and c the speed of the light. Taking into consideration Eq. (21), we can write $\phi_M = \frac{Gm}{r} [1 + \lambda_C^2 P_2(\cos \theta)/4r^2]$ and $\phi_E = \frac{kq}{r} [1 + g\lambda_C^2 P_2(\cos \theta)/8r^2]$ to note that, in this case, corrections are expected at lengths of the order of the reduced Compton wavelength and additionally the expressions between brackets for the modified potentials become identical for the case of the Kerr-Newman solution $g_{\text{KN}} = 2$. The modified Coulomb potential given above coincides with the potential given by Rosquist (*cf.* Eq. (14) in [19]), except for the missing g factor and a factor $1/3$.

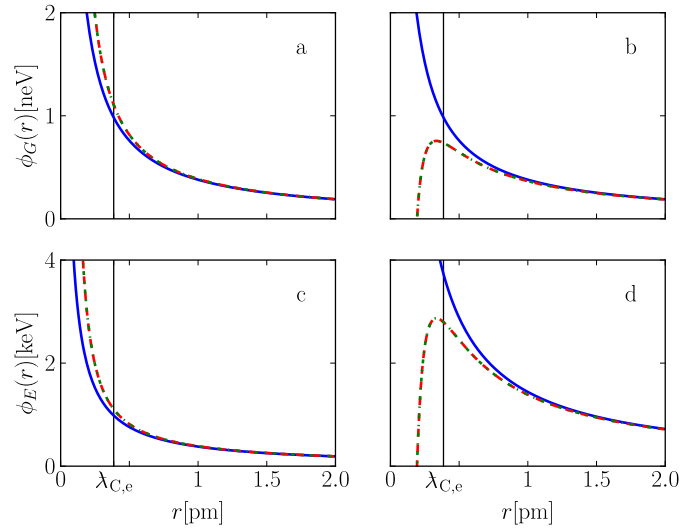


Figure 1. Gravitational [(a) and (b)] and electric [(c) and (d)] potential energy for the classical potentials (continuous line) together with the obtained by using Kerr-Newman's solution (dot-dashed line, $g_{\text{KN}} = 2$) and Manko's solution [dashed line, $g_e = 2.0023193043768(86)$] for the electron. Panels (a) and (c) stand for potentials on the equator while (b) and (d) do along the spin axis. $\lambda_{C,e}$ denotes the reduced Compton wavelength of the electron $\lambda_{C,e} = 0.386159$ pm.

In Figs. (1-2), we plot the classical gravitational and electric potential energies of a test particle with unit charge and mass together with the corrections predicted from the Kerr-Newman solution and from Manko's solution. In spite of the fact that the effect of using the real Lande g -factor becomes appreciable only for the electric potential of the proton at length scales below 1 fm [see Figs. (1-2)], in general, the gravitational and electric interaction will deviate significantly from its classical form at the reduced Compton scale. As can be seen from Figs. (1-2) the differences between the potential shapes along the spin axis and over the equatorial plane is not an effect introduced by the Lande g -factor but rather is a consequence of the quadrupolar-like symmetry of the Kerr-Newman and Manko solutions. For the gravitational potential, the quadrupolar mass term does not depend on g and is given by $m_2 = -a^2 m$ for both of the solutions; while for the electric potential, the

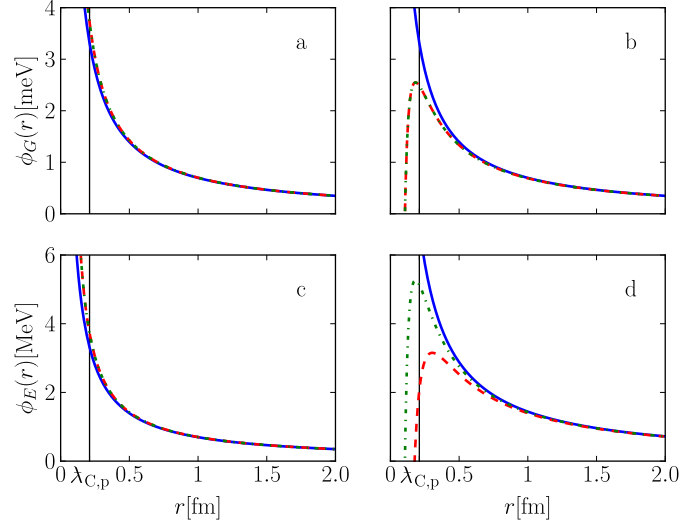


Figure 2. Gravitational [(a) and (b)] and electric [(c) and (d)] potential energy for the classical potentials (continuous line) together with the obtained by using Kerr-Newman' solution (dot-dashed line, $g_{\text{KN}} = 2$) and Manko's solution [dashed line, $g_{\text{P}} = 5.585694713(46)$] for the proton. Panels (a) and (c) stand for potentials on the equator while (b) and (d) do along the spin axis. $\lambda_{\text{C,p}}$ denotes the reduced Compton wavelength of the proton $\lambda_{\text{C,p}} = 0.210309$ fm.

quadrupolar term is given by $q_2^{\text{KN}} = -qJ^2/m^2$ for Kerr-Newman' solution and by $q_2 = -gqJ^2/2m^2$ for Manko's solution, so this will tend to be more marked for the proton.

Since at the Compton-wavelength scale the gravitational field becomes spin dominated rather than mass dominated, it is expected that some corrections to the hydrogen atom could be experimentally tested [19]. In order to render this argument more quantitative, we calculate the Bohr radius and the bound state energies of the electron for hydrogen-like atoms. Due to the fact that at distances of atomic radius the iso-potential surfaces in (22) are practically spherical, we take the expression over the equator. Under these assumptions, the explicit formulas are given by

$$R = \frac{c^2 n^2 \hbar^2 + \sqrt{c^4 n^4 \hbar^4 - 3c^2 \Omega}}{2km_e c^2 q^2 Z} \quad (23)$$

$$E = - \frac{2m_e k^2 c^4 q^4 Z^2 (c^2 n^4 \hbar^4 + n^2 \hbar^2 \sqrt{c^4 n^4 \hbar^4 - 3c^2 \Omega} - 2\Omega)}{(c^2 n^2 \hbar^2 + \sqrt{c^4 n^4 \hbar^4 - 3c^2 \Omega})^3} \quad (24)$$

where $\Omega = g_{\text{P}} a_{\text{P}}^2 k^2 m_{\text{e}}^2 q^4$, m_{e} is the electron mass, g_{P} the Lande g -factor for the proton, Z is the atomic number, n the principal quantum number and a_{P} is the angular momentum per unit mass for the proton. To get an idea of the order of the possible corrections, here we have assumed that the total nuclear angular momentum per mass unit is $a_Z = \hbar/(2Zm_{\text{P}})$; however, in real nuclei this value can certainly be completely different and additionally $m_Z \neq Zm_{\text{P}}$. Note that after neglecting the spin-dependent terms, *i.e.* in the limit $a_{\text{P}} \rightarrow 0$, expressions (23)-(24) reduce to the well known

classical formulas. The calculated energy and radius differences between the new expressions (23)-(24) and the predicted by the classical Bohr model for the first hydrogen-like atoms with $n = 1$ are listed below in Table 1.

	$\Delta E(10^{-10} \text{ eV})$	$\Delta R(10^{-12} \text{ Å})$
H	1.50047	8.75383
He ⁺	6.00188	4.37692
Li ²⁺	13.5043	2.91794
Be ³⁺	24.0075	2.18846
B ⁴⁺	37.5115	1.75076

Table 1. Differences between the new expressions (23)-(24) and the predicted by Bohr for the energy ΔE and the radius ΔR for the first hydrogen-like atoms in the ground state. Since the measurement uncertainty of the Bohr's radius is $3.6 \times 10^{-10} \text{ Å}$ [22] and the uncertainty for the Rydberg constant times hc is $3.4 \times 10^{-7} \text{ eV}$ [22], our corrections for these cases are just behind the uncertainty of the measurements of the fundamentals constants.

In spite of, for light atoms these estimations are not promising, based on the tendency, one could expect that possible corrections to the energy of the ground state of highly ionized heavy atoms could be tested, e.g., in the case of a molybdenum atom Mo⁴¹⁺ the correction to the energy would be of the order of $0.264685 \mu\text{eV}$ ($0.094773 \mu\text{eV}$ using g_{KN}) and for a gold atom Au⁷⁸⁺ the correction of the energy would be of the order of $0.936459 \mu\text{eV}$ ($0.335291 \mu\text{eV}$ using g_{KN}). Although reaching such setup is practically impossible for the gold because it would imply, e.g., tremendously high temperatures, there already exists data in Literature for H-like spectra of Mo¶, the binding energy for $n = 1$ of Mo⁴¹⁺ was estimated about $24572.21 \pm 0.01 \text{ eV}$ using QDE corrections [37]. Our estimation provides a value of $24000.4405842513 \text{ eV}$ while the usual Bohr expression leaves $24000.440584516 \text{ eV}$, both values differ from the QDE calculated-value, and our correction is some orders of magnitude below the uncertainty of the value predicted by QDE. However, it is important to mention that, in this case, the length scales at which this correction takes place, $R(\text{Mo}^{41+}) \sim 1.25995 \text{ pm}$, is far from the scale at which deviations are expected, $a_{\text{Mo}}/c = \hbar/(2cZm_p) \sim 8.35 \times 10^{-15} \text{ pm}$.

Finally, we want to make contact with previous results from the one-loop quantum corrections to the Newton and Coulomb potential induced by the combination of graviton and photon fluctuations. In particular, we choose the recent calculations in [6, 7] considering massive charged spin- $\frac{1}{2}$ fermions. This calculation already contains relativistic corrections, the leading order corrections are $V_{\text{pN}} = Gm/c^2 r$ for the relativistic post-Newtonian term and $V_q = G\hbar/c^3 r^2 = l_p^2/r^2$ for the quantum corrections, however, they are by some order of magnitudes smaller than our leading corrections: $ga^2/2c^2 r^2$ for the Coulomb potential and $a^2/c^2 r^2$ for the Newtonian potential in (22). At the order of the reduced Compton wavelength, V_{pN} and V_q are of the order of 10^{-39} for the proton and 10^{-45} for the electron, while (22) predicts corrections of the order of 10^{-1} . This can be appreciated in Figs. (1-2) around the reduced Compton wavelength of the electron $\lambda_{\text{C,e}}$ and of the proton $\lambda_{\text{C,p}}$. Since our corrections come from the rotation of the source, this suggests that angular momentum

¶ Additional data for H-like spectra of Ti, V, Cr, Mn, Fe, Co, Ni, Cu, and Kr are also available [37] and some analytic quantum estimations for Rb have been given in [38].

is worth considering in quantum gravity calculations in a more detailed way.

5. Concluding remarks

We have studied the influence of the gyromagnetic ratio in the description of the classical fields of the electron and proton, and showed that the description based on Kerr-Newman solution deviates significantly for the proton. Based on our assumptions in Sec. 4, we have obtained that, although general relativistic effects could be expected in highly ionized heavy atoms, our estimations could hardly be detected. However, we consider that a more detailed analysis should be carried out, e.g. taking into account the non-central character of the corrections and considering nuclei with a higher angular momentum, e.g. in $^{95}_{42}\text{Mo}$ (spin 5/2), $^{73}_{32}\text{Ge}$ (spin 9/2) or $^{195\text{m}}_{80}\text{Hg}$ (spin 13/2) or explore, e.g., if our corrections would change the hyperfine splitting of hydrogen. On the other hand, we also point out the necessity of including angular momentum effects in the quantum description of gravity at the order of at the Compton length scale.

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- [1] Epstein H and Glaser V. *Annales Poincare Phys. Theor.*, A19:211, 1973.
- [2] Helayël-Neto J A, Penna-Firme A B, and Shapiro I L. *JHEP*, 01:009, 2000.
- [3] Elizalde E, Lousto C O, Odintsov S D, and Romeo A. *Phys. Rev. D*, 52:2202, 1995.
- [4] Grillo N. *Class. Quantum Grav.*, 18:141, 2001.
- [5] Bjerrum-Bohr N E J. *Phys. Rev. D*, 66:084023, 2002.
- [6] Bjerrum-Bohr N E J, Donoghue J F, and Holstein B R. *Phys. Rev. D*, 67:084033, 2003.
- [7] Butt M S. *Phys. Rev. D*, 74:125007, 2006.
- [8] Faller S. *Phys. Rev. D*, 77:124039, 2008.
- [9] Martin A W and Pritchett P L. *J. Math. Phys.*, 9:593, 1968.
- [10] Israel W. *Phys. Rev. D*, 2:641, 1970.
- [11] Newman E T and Winicour J. *J. Math. Phys.*, 15:1113, 1974.
- [12] Burinskii A Ya. *Sov. Phys. JETP*, 39:193, 1974.
- [13] López C A. *Phys. Rev. D*, 30:313, 1984.
- [14] Mann R B and Morris M S. *Phys. Lett. A*, 181:443, 1993.
- [15] Burinskii A. *Phys. Rev. D*, 68:105004, 2003.
- [16] Arcos H I and Pereira J G. *Gen. Relativ. Gravit.*, 36:2441, 2004.
- [17] Burinskii A. *Phys. Rev. D*, 70:086006, 2004.
- [18] Burinskii A. *Czech. J. Phys.*, 55:A261, 2005.
- [19] Rosquist K. *Class. Quantum Grav.*, 23:3111, 2006.
- [20] Carter B. *Phys. Rev.*, 174:1559, 1968.
- [21] Newman E T, Couch E, Chinnapared K, Exton A, Prakash A, and Torrence R. *J. Math. Phys.*, 6:918, 1965.
- [22] Mohr P J, Taylor B N, and Newell D B. *Rev. Mod. Phys.*, 80(2):633, 2008.
- [23] Manko V S. *Phys. Lett. A*, 181:349, 1993.

- [24] Ernst F J. *Phys. Rev.*, 168:1415, 1968.
- [25] Manko V S and Sibgatulli N R. *Class. Quantum Grav.*, 10:1383, 1993.
- [26] Sibgatullin N R. *Oscillations and waves in strong gravitational and electromagnetic fields (Engl. transl.)*. Springer, Berlin [orig. Russian, 1984, Nauka, Moscow], 1991.
- [27] Sotiriou P and Apostolatos A. *Class. Quantum Grav.*, 21:5727, 2004.
- [28] Geroch R. *J. Math. Phys.*, 11:2580, 1970.
- [29] Hansen R O. *J. Math. Phys.*, 15:46, 1974.
- [30] Pachón L A and Sanabria-Gómez J D. *Class. Quantum Grav.*, 23:3251, 2006.
- [31] Feynman R P, Leighton R B, and Sands M. *The Feynman Lectures on Physics, Quantum Mechanics, Vol III*. Addison Wesley Publishing Company, Massachusetts, U.S.A., 1965.
- [32] Manko V S, Martín J, and Ruíz E. *Phys. Rev. D*, 36:3063, 1995.
- [33] Pachón L A, Rueda J A, and Sanabria-Gómez J D. *Phys. Rev. D*, 73:104038, 2006.
- [34] Bonnor W B. *Phys. Lett. A*, 158:23, 1991.
- [35] Bonnor W B. *Gen. Relativ. Gravit.*, 38:1063, 2006.
- [36] Herrera L, González G A, Pachón L A, and Rueda J A. *Class. Quantum Grav.*, 23:2395, 2006.
- [37] Shirai T, Sugar J, Musgrove A, and Wiese W L. *Spectral Data For Highly Ionized Atoms: Ti, V, Cr, Mn, Fe, Co, Ni, Cu, Kr, and Mo*. J. Phys. Chem. Ref. Data, Monograph No. 8, 2000.
- [38] Sansonetti J E. *J. Phys. Chem. Ref. Data*, 35:301, 2006.